Correlation functions in chaotic systems from periodic orbits

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We express correlation functions in chaotic systems as averages over correlation functions along periodic orbits and use the thermodynamic formalism and the cycle expansion to obtain high precision results.

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I. INTRODUCTION

The correlation function $C_{AB}(t)$ between two observables $A(\mathbf{x})$ and $B(\mathbf{x})$ along a trajectory $\{\mathbf{x}(t)\}$ is defined as the average,

$$C_{AB}(t) = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(\mathbf{x}(\tau+t))\, B(\mathbf{x}(\tau))\, d\tau\,. \quad (1)$$

If the dynamics is ergodic, almost all trajectories give the same correlation function. Then $C_{AB}(t)$ may also be obtained from the invariant measure $d\mu(\mathbf{x})$,

$$C_{AB}(t) = \int A(\mathbf{x}(t;\mathbf{x}_0))\, B(\mathbf{x}_0)\, d\mu(\mathbf{x}_0)\,, \quad (2)$$

where now $\mathbf{x}(t;\mathbf{x}_0)$ denotes the point reached after a time t by a trajectory starting from $\mathbf{x}_0$. In chaotic systems with few degrees of freedom, Eq. (1) is the form most useful in practice since it can be applied directly to time series obtained from experiment or computation. The second form is usually less useful because the invariant measure, often a nonuniform distribution supported by a fractal set, is rather difficult to characterize [1].

Some progress can be made if the dynamics is hyperbolic. Then periodic orbits are dense in the invariant set and can be used to approximate the measure [1–4]. zeta functions and the cycle expansion can be used to tame the exponential proliferation of orbits, thus allowing accurate computations [5–7]. This technique has been applied to computations of spectra in maps and flows [8–10]. Here, we extend the formalism to obtain full correlation functions. We will mainly focus on maps, but the formulas for time continuous systems will be listed as well.

The outline of the paper is as follows. In the next section, we discuss the thermodynamic formalism for the calculation of averages from periodic orbits, the specific form for correlation functions, the extension to time continuous systems, and some computational aspects. In Sec. III we apply the formalism to the tent map and a perturbed tent map. We conclude with some final remarks in Sec. IV.

II. THERMODYNAMIC FORMALISM FOR AVERAGES

As Eq. (2) suggests, $C_{AB}(t)$ emerges as the average of the quantity $a(\mathbf{x}_0) = A(\mathbf{x}(t;\mathbf{x}_0))B(\mathbf{x}_0)$ over the invariant measure. Thus, correlation functions may be obtained from phase space averages of a special kind of observable. We, therefore, treat the general formalism for averages first before specializing to correlation functions.

A. Generalized evolution operator

To relate phase space averages of any observable $a(\mathbf{x})$ to periodic orbits, consider the generalized evolution operator [11,12],

$$\mathcal{L}_q(\mathbf{y},\mathbf{x}) = \delta(\mathbf{y}-\mathbf{f}(\mathbf{x}))e^{qa(\mathbf{x})}, \quad (3)$$

and its nth "iterate,"

$$\begin{aligned}\mathcal{L}_q^{(n)}(\mathbf{y},\mathbf{x}) &= \int d\mathbf{x}_{n-1}\cdots d\mathbf{x}_1 \mathcal{L}_q(\mathbf{y},\mathbf{x}_{n-1}) \\ &\quad\times \prod_{k=2}^{n-1} \mathcal{L}_q(\mathbf{x}_k,\mathbf{x}_{k-1})\ \mathcal{L}_q(\mathbf{x}_1,\mathbf{x}) \\ &= \delta[\mathbf{y}-\mathbf{f}^{(n)}(\mathbf{x})] \exp\left(q\sum_{k=0}^{n-1} a(\mathbf{f}^{(k)}(\mathbf{x}))\right). \end{aligned} \quad (4)$$

This evolution operator carries information about both the discrete time dynamics and the observable of interest. Periodic orbits are selected by taking the trace of $\mathcal{L}_q^{(n)}$,

$$\begin{aligned}\mathrm{tr}\,\mathcal{L}_q^{(n)} &= \int d\mathbf{x}\, \delta[\mathbf{x}-\mathbf{f}^{(n)}(\mathbf{x})] \exp\left(q\sum_{k=0}^{n-1} a(\mathbf{f}^{(k)}(\mathbf{x}))\right) \\ &= \sum_{\mathbf{x}_P^{(n)}} w(\mathbf{x}_P^{(n)}) e^{qa(\mathbf{x}_P^{(n)})}, \end{aligned} \quad (5)$$

where $\mathbf{x}_P^{(n)}$ are the points periodic after n iterations, and where

$$a(\mathbf{x}_P^{(n)}) = a_P = \sum_{k=0}^{n-1} a(\mathbf{f}^{(k)}(\mathbf{x}_P^{(n)})), \tag{6}$$

is the sum of the observable along the trajectory starting at $\mathbf{x}_P^{(n)}$. The weights of the periodic points are given by

$$w(\mathbf{x}_P^{(n)}) = w_P = \left|\det[1 - D\mathbf{f}^{(n)}(\mathbf{x}_P^{(n)})]\right|^{-1}, \tag{7}$$

with the derivative matrix

$$D\mathbf{f}^{(n)}(\mathbf{x}_P^{(n)}) = \prod_{k=0}^{n-1} D\mathbf{f}(\mathbf{f}^{(k)}(\mathbf{x}_P^{(n)})), \tag{8}$$

the product of the derivatives $D\mathbf{f}$ of the map along the orbit.

Within the thermodynamic formalism [4,13], one views this trace (5) as a kind of canonical partition function. The average of the a_P's may then be obtained as the logarithmic derivative of a suitable generating function. The average $\langle a\rangle_n$ of the observable a computed from all periodic orbits of period n is one nth this value, since the a_P's include a summation over all points of the trajectory. Thus,

$$\langle a\rangle_n = \frac{1}{n}\frac{\sum_{\mathbf{x}_P^{(n)}} w_P a_P}{\sum_{\mathbf{x}_P^{(n)}} w_P} = \frac{1}{n}\frac{d}{dq}\ln \mathrm{tr}\,\mathcal{L}_q^{(n)}\Big|_{q=0}. \tag{9}$$

Asymptotically, for diverging period n, we expect $\langle a\rangle_n$ to approach the average $\bar{a}$ of a over all of phase space and thus an exponential behavior of the trace, viz.

$$\mathrm{tr}\,\mathcal{L}_q^{(n)} \sim e^{\bar{a}nq} \qquad \text{for } n\to\infty. \tag{10}$$

This asymptotic behavior is most easily extracted from the poles of the grand canonical partition function,

$$\Omega_q(z) = \sum_{n=1}^{\infty} z^n \mathrm{tr}\,\mathcal{L}_q^{(n)}. \tag{11}$$

Its leading pole behaves like $z_0(q) \sim e^{-q\bar{a}}$, so that the average is given by

$$\bar{a} = -\frac{d}{dq}\ln z_0(q)|_{q=0}. \tag{12}$$

This formula simplifies slightly if we take into account that the leading pole for a hyperbolic map is $z_0(q=0)=1$, whence,

$$\bar{a} = -\frac{d}{dq} z_0(q)|_{q=0}. \tag{13}$$

In principle, one could compute $\bar{a}$ from these averages directly. However, one can improve on this by going to the ζ functions [6,7,14].

B. zeta function

Using the periodic orbit expression for the trace of $\mathcal{L}_q$, Eq. (5), a zeta function representation for the grand canonical partition function can be derived [6]. Note that for all points along a periodic trajectory, w_P and a_P take on the same values, since only the order of summation in (6) and the order of products in (8) change when shifting the initial point. Moreover, if a periodic orbit is traversed r times, these quantities become multiples and powers of the ones for a single traversal, respectively.

Thus, replacing the summations in (11) by sums over all primitive periodic orbits p with n_p different points, the derivative matrix D_p [Eq. (8)] for the primitive periodic orbit (raised to the power r), and summed observable a_p, we find [6]

$$\Omega_q(z) = \sum_p \sum_{r=1}^{\infty} \frac{n_p z^{n_p r} e^{q a_p r}}{|\det[1-(D_p)^r]|}. \tag{14}$$

We now define the zeta function $Z_q(z)$ such that its negative logarithmic derivative with respect to z equals $\Omega_q(z)$,

$$Z_q(z) = \exp\left\{-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{z^{n_p r}e^{q a_p r}}{|\det(1-(D_p)^r)|}\right\}. \tag{15}$$

For $q=0$, this zeta function becomes the Fredholm determinant of the evolution operator [15]. By the usual expansions [6], this zeta function can also be written as an infinite product over periodic orbits. But for most purposes, including numerical calculations, this form is satisfactory (see Sec. II E).

C. Correlation functions

Any observable a can enter in the generalized evolution operator (3). Correlation functions are obtained for the choice $a(\mathbf{x}_0) = A(\mathbf{x}(t;\mathbf{x}_0))B(\mathbf{x}_0)$. Specifically, for maps $a(\mathbf{x}) = A(\mathbf{f}^t(\mathbf{x}))B(\mathbf{x})$ and, thus,

$$a_P = \sum_{k=0}^{n_P-1} A(\mathbf{f}^{(k+t)}(\mathbf{x}_P))B(\mathbf{f}^{(k)}(\mathbf{x}_P)), \tag{16}$$

where $\mathbf{x}_P$ is a point on the periodic orbit P. Except for a division through the period n_P of the orbit this expression equals the correlation function (1), for A and B along the periodic orbit. As one might have anticipated, the canonical formula (9) thus expresses the ergodic correlation function as a weighted average of correlation functions along periodic orbits. The grand canonical partition function and, in particular, the zeta function also lead to a weighted average over periodic orbit correlation functions, but now the interpretation is less obvious.

D. Time continuous systems

For time continuous systems the main difference is that discrete time n has to be replaced by continuous time t [16]. Taking the trace as in Eq. (5) at some time t, one now finds a contribution only if there is a periodic orbit with primitive period $T_p = t$ or a multiply traversed shorter orbit with $rT_p = t$. The weight of such an orbit can be evaluated by a transformation to a coordinate system with one coordinate parallel to the trajectory and all others perpendicular. If M_p denotes the linearization of the map in a plane perpendicular to the trajectory, then the weight of a primitive periodic orbit is given by

$$w_p = |\det(1 - M_p)|^{-1}, \tag{17}$$

and the sum of the observable over the trajectory is replaced by

$$a_p = \int_0^{T_p} a(\mathbf{x}(t))\, dt. \tag{18}$$

The grand canonical partition function is now obtained from an integral over all times,

$$\begin{aligned}\Omega_q(s) &= \int_0^\infty e^{-st} \mathrm{tr}\, \mathcal{L}_q^{(t)}\, dt \\ &= \sum_p \sum_{r=1}^\infty \frac{T_p e^{-sT_p r} e^{q a_p r}}{|\det[1 - (M_p)^r]|}.\end{aligned} \tag{19}$$

In the case of discrete maps (Sec. II B), the next steps were to write this as a logarithmic derivative of a zeta function and to expand the zeta function. This was done in the variable z, counting the discrete periods of the orbits. For time continuous systems the spectrum of periods is still discrete but usually not quantized to integer multiples of a fixed period, unless the system is periodic. However, a quantized "length" of orbits may always be introduced by counting the number of crossings of a Poincaré surface of section. In addition, this surface of section may be used, very much as in the discrete time case, to introduce a symbolic coding for trajectories. We, thus, extend the above expression by including in each contribution the term $z^{n_p r}$, where n_p is the number of crossings of a surface of section or the number of symbols in the primitive periodic orbit and where r is the number of traversals of this primitive periodic orbit. This variable z is used for the expansion only and is set to one in the final calculations. We then write the grand canonical partition function as a logarithmic dervivative with respect to s,

$$\Omega_q(s) = \frac{d}{ds} \ln Z_q(s,z) \Big|_{z=1}, \tag{20}$$

where the zeta function is given by

$$Z_q(s,z) = \exp\left\{ -\sum_p \sum_{r=1}^\infty \frac{z^{n_p r}}{r} \frac{e^{-sT_p r} e^{q a_p r}}{|\det[1 - (M_p)^r]|} \right\}. \tag{21}$$

In actual calculations, one expands this expression in powers of z, keeping again only terms with powers up to the largest period (number of crossings) N for which all periodic orbits are known. The averages we are interested in now emerges as the derivative of the leading zero $s_0(q, z = 1)$ with respect to q at $q = 0$,

$$\bar{a} = \frac{d}{dq} s_0(q, z = 1) \Big|_{q=0}. \tag{22}$$

There is no logarithm here since s has to be identified with $-\ln z$ when compared to the discrete dynamics.

E. Computational aspects

The calculation of periodic orbits required for this formalism proceeds most conveniently with a multipoint shooting method, perhaps combined with an adiabatic following technique if periodic points cluster in some regions of phase space. Details are described in Refs. [10,17].

To evaluate the zeta function, one most conveniently first computes the coefficients b_k in the representation,

$$Z_q(z) = \exp\left(-\sum_{k=1}^\infty b_k z^k \right), \tag{23}$$

and then uses the recursion relation of Plemelj and Smithies [18–20] to compute the approximate power series expansion,

$$\exp\left(-\sum_{k=1}^{n_{\max}} b_k z^k \right) \approx \sum_{j=0}^{n_{\max}} c_j z^j. \tag{24}$$

$n_{\max}$ denotes both the largest period for which all periodic points are known as well as the highest power to which the expansion can consistently be computed since the coefficients c_j depend on all b_k with $k \le j$. In cases where the zeta function is entire, the higher order coefficients decay faster than exponential (see, e.g., [21]), thus allowing for efficient computation of the leading zero which contains the desired information.

III. APPLICATION TO TENT MAPS

A. General features

Shifted tent maps on the unit interval are among the simplest hyperbolic systems [22,23]. Using a parametrization by the shift s of the tip of the tent from its symmetric position at $x = 1/2$, they are given by

$$f(x) = \begin{cases} 2x/(1+2s), & 0 \le x \le \frac{1}{2} + s \\ (2-2x)/(1-2s), & \frac{1}{2} + s \le x \le 1. \end{cases} \tag{25}$$

Symbols L and R may be introduced according to a trajectory point $\mathbf{x}_t$ visiting the intervals to the left or right of the maximum. This encoding is complete, there being a one-to-one correspondence between periodic symbol strings and periodic orbits. The fixed points are $x_L = 0$ with derivative $D_L = 2/(1+2s)$ and $x_R = 2/(3-2s)$ with derivative $D_R = -2/(1-2s)$. Because of the uniformity of the derivatives in each interval, derivatives for longer orbits factorized into products of powers of D_L and D_R, specifically, $D_p = D_L^{n_L} D_R^{n_R}$, where n_L and n_R with $n_R + n_L = n_p$ are the numbers of L and R symbols in the string p.

The Fredholm determinants for this system can easily be computed and factorized according to [6,9]

$$Z_{q=0}(z) = \prod_{j=0}^\infty \left[1 - z\left(\frac{1}{|D_L| D_L^j} + \frac{1}{|D_R| D_R^j} \right) \right]. \tag{26}$$

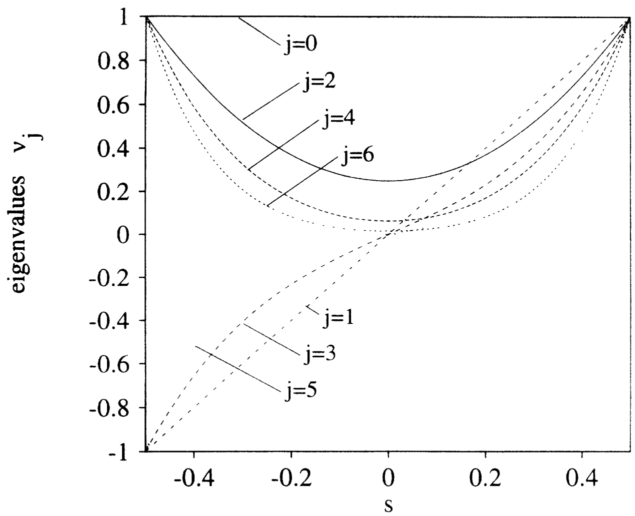

FIG. 1. Eigenvalues of the tent map (25) for different values of the shift s and for $j = 1, \ldots, 6$. Note that $\nu_0 = 1$. Eigenvalues for odd j cross through the zero for $s = 0$ and approach -1 for $s \to -1/2$ and $+1$ for $s \to 1/2$. Eigenvalues for even j are always positive and approach 1 at the boundaries of the interval. The eigenvalues for $j = 1$ and 2 cross at $s = 1/6$.

The zeros of this expression are the inverses of the eigenvalues ν_j of the evolution operator. As a function of the shift s, the largest ones read

$$\begin{aligned}\nu_0 &= |D_L|^{-1} + |D_R|^{-1} = 1,\\ \nu_1 &= |D_L|^{-1}D_L^{-1} + |D_R|^{-1}D_R^{-1} = 2s,\\ \nu_2 &= |D_L|^{-1}D_L^{-2} + |D_R|^{-1}D_R^{-2} = 1/4 + 3s^2,\\ \nu_3 &= |D_L|^{-1}D_L^{-3} + |D_R|^{-1}D_R^{-3} = s + 4s^3, \end{aligned} \quad (27)$$

and so forth. They are displayed Fig. 1. One notes degeneracies of eigenvalues, for example, at $s = 1/6$, where $\nu_1 = \nu_2 = 1/3$. The degeneracies show up in correlation functions in the form of a $t\nu^{-t}$ behavior. The infinite degeneracy at $s = 0$ of all odd resonances is compensated by the fact that all frequencies vanish, $\nu_1 = \nu_3 = \nu_5 = \cdots = 0$, and, thus, no pecularities arise.

B. Correlation functions

We have computed correlation functions for several observables and several parameter values. As a typical example, we pick the observable $\sin x$. Values for the average $\langle \sin x \rangle$ and for the autocorrelation of $A(x) = B(x) = \sin x - \langle \sin x \rangle$ for different time shifts as obtained from periodic orbits up to period 13 are listed in Table I. One notes the rapid decay of the correlations for $s = 0$ and the loss of significant digits as the time parameter in the correlation function is increased. In our calculation using periodic orbits up to period 13, no significant digits are left beyond $t = 9$.

It is known that the cycle expansion for the Fredholm determinant for the usual Frobenius Perron operator converges faster than exponential [15,24,21]. Our numerical results suggest that this is also the case for the generalized evolution operator. This is demonstrated in Fig. 2 for the observable $\sin x$ for $s = 0$ and different values of t. Clearly, the convergence cannot set in until the period of the orbits included exceeds t, since a longer time delay probes correlations which can only be contained in orbits of sufficiently large periods.

As the parameter s deviates from zero, convergence slows down. Clearly, as $s \to \pm 1/2$, one fixed point is only marginally unstable and hyperbolicity is lost. Also, in the spectrum, all eigenvalues cluster at ± 1, causing a slower and perhaps no longer exponential decay in correlations. As shown in Table I, for $s = +0.2$ correlations could be determined up to $t = 6$, and up to $t = 7$ for $s = -0.2$. For negative s, the alternations in sign of the correlations

TABLE I. Average and autocorrelation function for the variables $A(x) = B(x) = \sin x - \langle \sin x \rangle$ for the tent map (25) for three values of the parameter s. Periodic orbits up to period 13 are used. Listed are those digits which agreed in calculations using periodic orbits up to periods 12 and 13, respectively. The average does not depend on s since the invariant density is the same constant, namely 1, in all cases. Thus the number of significant digits listed is a first indication of the accuracy that can possibly be achieved for correlation functions.

	$s = 0$	$s = 0.2$	$s = -0.2$
$\langle \sin x \rangle$	0.45969 76941 32	0.45969 7693	0.45969 76941 33
$t = 0$	0.06135 36733 03	0.06135 3673	0.06135 36733 03
1	0.00427 90545 48	0.02829 2220	−0.02101 31011 88
2	0.00104 98206 94	0.01239 8026	0.01013 10110 31
3	0.00026 12382 57	0.00528 2112	−0.00350 72419 87
4	0.00006 52339 57	0.00221 37	0.00162 72066
5	0.00001 63037 71	0.00091 8	−0.00057 2997
6	0.00000 40756 48	0.0004	0.00025 92
7	0.00000 10188 9		−0.00009
8	0.00000 025		
9	0.00000 015		

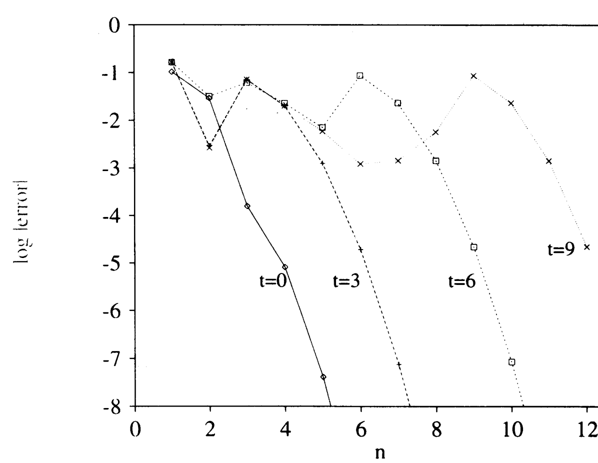

FIG. 2. Convergence of the autocorrelation function for $\sin x$ for $s = 0$ and different time delays $t = 0, 3, 6, 9$ (Table I). Shown is the decadic logarithm of the error (difference between current and exact value) vs the highest period n included. All autocorrelation functions converge rapidly and faster than exponential once the period n of the orbits included exceeds the delay t.

TABLE II. Average and correlation function for the variables $A(x) = x$ and $(x-1)^2$ for the tent maps (25) at $s = 1/6$. Significant figures were estimated from zeta function results with periods 12 and 13. Within this accuracy, the data are compatible with the analytical formulas as given in the last line.

	$A(x) = x$	$A(x) = (x-1)^2$
$\langle A \rangle$	0.5	0.33333 33333 33
$t = 0$	0.08333 33333 33	0.08888 88888 88
1	0.02777 77777 78	0.04814 81481 48
2	0.00925 92592 59	0.02222 22222 22
3	0.00308 64197	0.00946 50205
4	0.00102 88065	0.00384 08782
5	0.00034 294	0.00150 89
6	0.00011	0.00057 9
7	0.00004	0.0002
t	$3^{-t}/12$	$(8+5t)3^{-t}/90$

and the eigenvalues seem to help convergence a bit.

As mentioned in section III A, there is a degeneracy of eigenvalues at $s = 1/6$. To test for the linear modification of the exponential decay, we have computed correlation functions for the observables x and $(x-1)^2$ (minus their averages). In Table II we list the digits that agree in approximations using orbits up to period 12 and 13, respectively. The data are compatible with a pure exponential behavior $C(t) = 3^{-t}/12$ for the linear variable and $C(t) = (8+5t)3^{-t}/90$ for the quadratic variable.

All calculations here could, in principle, be done analytically, using a matrix representation of the evolution operator acting on powers of x [23]. However, the advantage of the periodic orbit formalism is that it carries over without change to perturbations of the linear map (25).

C. A perturbed shift map

Christiansen *et al.* [8] in their study of the spectra of maps used an asymmetrically perturbed shift map on the interval $[0,1]$,

$$f(x) = \begin{cases} -3.5 + 5\sqrt{0.4x + 0.49}, & 0 \le x \le 0.8 \\ -0.5 + 5\sqrt{0.4x - 0.31}, & 0.8 \le x \le 1. \end{cases} \quad (28)$$

In Table III, we present the correlation function of the fluctuations of the observable $A(x) = x$ around its mean. Convergence is as good as in the previous case, thus demonstrating the ease of use for perturbed maps.

TABLE III. Correlation function for the variable $A(x) = x - \langle x \rangle$ for the perturbed shift map (28). The average of the observable is 0.5 with an accuracy of 10^{-12}.

t	$C(t)$
0	0.08333 33334
1	0.04166 66669
2	0.01628 33339
3	0.00336 00038
4	−0.00166 507
5	−0.00262 49
6	−0.00202 92
7	−0.0012

IV. CONCLUSIONS

We have expressed the classical correlation function as a weighted average over correlation functions along periodic orbits. The calculations show that rather rapid convergence sets in once the period of the orbits included exceeds the time delay in the correlation function. Qualitatively, the rate of convergence is controlled by the resonance nearest to the eigenvalue 1.

Applications of the above formalism to many other hyperbolic systems are rather straightforward. For nonhyperbolic systems one encounters the usual problems [6,7,14]. If the symbolic dynamics is not generated by a subshift of finite type, one cannot expect faster than exponential convergence [15]. And if there are marginally stable orbits with $|\Lambda| = 1$ or close to one, the zeta function is not entire and the convergence is very slow, perhaps algebraic rather than exponential. In principle, the methods of Artuso *et al.* [25] for marginally stable orbits should be applicable, but they will become cumbersome in practice because of the exponential proliferation of the number of periodic orbits with period. Besides these technical points, there is no difference in the formalism.

The above calculations also shed some light on similar expressions in semiclassical mechanics. It is gratifying to note that just as in the case of the spectrum, there is a similar expression within the semiclassical approximation for quantum systems [16,26,27]. The main difference lies in the weights: whereas classically one adds probabilities, quantum mechanically one has to add amplitudes, thus introducing the square root of the classical weight and phase factors due to caustics. Of course, the quantum observables also have to be represented by suitable Wigner transformed phase space densities. The present calculations show that the periodic orbit formulas given in [27] can, in principle, be summed to yield the correlation functions. Unfortunately, there is no simple quantum system in which this could as yet be done.

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